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Dependence of $\hbar\omega$ on the Mass Number of Nuclei under the Assumption of a Trapezoidal Average Nuclear Density

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The variation of the harmonic oscillator energy level spacing $\hbar\omega$ with the mass number of nuclei is investigated under the assumption of a trapezoidal distribution for the average nuclear density and the results are compared with those obtained with a Fermi distribution. The dependence of the parameters on the mass number is also discussed.

1. INTRODUCTION

The dependence of the harmonic oscillator energy level spacing $\hbar\omega$ on the mass number A of nuclei has long been known (Moszkowski, 1957; De Shalit and Feshbach, 1974) to be of the form, for large A, $\hbar\omega = fA^{-1/3}$, where $f = \frac{5}{4}(\hbar^2/Mr_0^2)(\frac{3}{2})^{1/3} \simeq 41$ MeV ($r_0 \simeq 1.2$ fm). The proportionality constant may be also connected in an approximate way to the well depth V_0 (Goeppert Mayer et al., 1955). Various improvements to the above asymptotic law have been proposed, with the aim of obtaining expressions more satisfactory for lighter nuclei (Eder, 1968; Blomqvist and Molinari, 1968; Bertsch, 1972; H. Bethe, personal communication in Negele, 1970).

The problem of the A dependence of $\hbar\omega$ has been discussed in more recent investigations (Van Hees, 1982; Van Hees and Glaudemans, 1983; Daskaloyannis et al., 1983 and in preparation).

The analysis of Van Hees (1982) [the result is quoted in Van Hees and Glaudemans (1983)] was made by using radii of nuclei in the p shell and it was realized that the above asymptotic law is not valid in this region. It was found that a constant value of $\hbar\omega$ (independent of A) gives a better fit. The best fit value was ~ 13 MeV, but a small increase in this value results

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if the finite-proton-size correction is taken into account, the new best fit value being about 14.6 MeV. The realization that a constant value of $\hbar\omega$ through the p shell gives a good fit to charge radii is in agreement with the findings of Gal et al. (1971), though the value of $\hbar\omega$ given there is about 17 MeV.

Daskaloyannis et al. (1983 and in preparation) considered the possibility of nuclei with valence nucleons in the way suggested by the initial formulation of Hornyak (1975, p. 240), but without the use of an approximation in relating the number of the highest filled shell K to the mass number [see also Gunye and Warke (1970) in connection with this matter]. The center-of-mass and finite-size corrections were also taken into account in the usual way. Furthermore, instead of using for the average nuclear mean square radius the standard result for heavy nuclei, namely $\langle r^2\rangle = \frac{3}{5}r_0A^{2/3}$, the more satisfactory one corresponding to a Fermi distribution was employed (Daskaloyannis et al. 1983 and in preparation). This analysis led to an approximate expression for $\hbar\omega$, the corresponding "curve" of which as a function of A has "discontinuities in the slope" at the closed shells. In addition, a simplified expression for $\hbar\omega$ leading to a "smooth curve" was derived by keeping the two leading terms in the expansion of $\hbar\omega$ in powers of A.

The object of this paper is to investigate the variation of $\hbar\omega$ with A under the assumption of another type of average nuclear density, namely a trapezoidal one. This distribution, though less realistic than the Fermi one, has the advantage that the corresponding normalization integral and the one for the mean square radius can be obtained analytically in an exact way. For the Fermi distribution the expressions for the normalization integral and mean square radius used in practice are approximate. They are, however, very accurate apart from the lightest nuclei.

In the following section the expressions for the mean square radius (and its square root) and of the corresponding one for $\hbar\omega$ together with their expansions in powers of the mass number are considered and a comparison is made with those pertaining to the Fermi distribution. Furthermore, a possible generalization is discussed. In the final section, numerical values of the parameters determined by fitting to experimental values of rms radii of nuclei or to the values of $\hbar\omega$ corresponding to these radii are given, and the variation of $\hbar\omega$ with A obtained with these values is discussed.

2. THE EXPRESSIONS FOR $\langle r^2\rangle^{1/2}$, $\langle r^2\rangle$, AND $\hbar\omega$ AND THEIR EXPANSIONS IN POWERS OF THE MASS NUMBER

We proceed as in Daskaloyannis et al. (1983 and in preparation) by considering neutrons and protons as identical. We recall first that for the trapezoidal distribution (see Figure 1)

$$\rho_{\mathrm{tr}} = \begin{cases} \rho_0 & 0 < r < c - z \\ \dfrac{\rho_0(c + z - r)}{2z}, & c - z < r < c + z \\ 0, & c + z < r < \infty \end{cases} \tag{1}$$

the normalization condition

$$4\pi \int_0^\infty \rho_{\mathrm{tr}} r^2 \, dr = A \tag{2}$$

leads to the well-known expression (Hornyak, 1975, p. 115)

$$\rho_0 = \frac{3A}{4\pi c^3(1 + z^2/c^2)} \tag{3}$$

As was pointed out in Kodama (1971), the resulting third-order equation for the half-way radius c can be solved analytically. The result can be put in the form

$$c = (\tfrac{1}{2})^{1/3} bA^{1/3}(\{1 + [1 + 2^2 \cdot 3^{-3}(z/bA^{1/3})^6]^{1/2}\}^{1/3} + \{1 - [1 + 2^2 \cdot 3^{-3}(z/bA^{1/3})^6]^{1/2}\}^{1/3}) \tag{4}$$

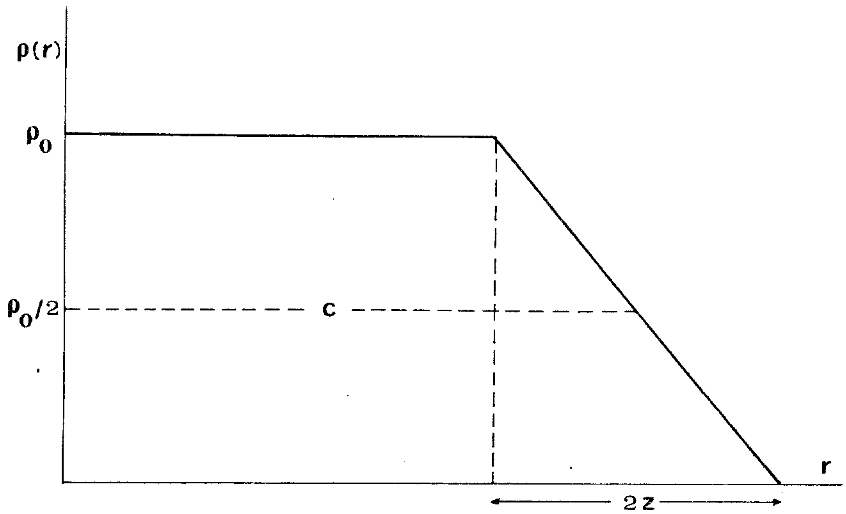

Fig. 1. The trapezoidal distribution.

where $b=(3/4\pi\rho_0)^{1/3}$ and $2z$ is the width of the surface region. Expression (4) was also used in Uno and Yamada (1975) in connection with the mass formula. The corresponding expression for c for the Fermi distribution

$$\rho_F = \frac{\rho_0}{1+e^{(r-c)/\alpha}} \tag{5}$$

is formally the same with the above one (only $\pi\alpha$ has to be replaced by z), but it is not exact. It is valid if $e^{-c/\alpha} \ll 1$, that is, provided the nucleus is not very light.

The dependence of the rms radius on the mass number is immediately established by using expression (4) for c and the expression for $\langle r^2 \rangle$ given in Hornyak (1975) (in which z corresponds to $R_2\delta$):

$$\langle r^2\rangle_{tr} = \tfrac{3}{5} b^{-3} A^{-1} c(c^2+\tfrac{1}{3}z^2)(c^2+3z^2) = \tfrac{3}{5}(Ab^3)^{-1} c^3(c^2+\tfrac{10}{3}z^2+z^4c^{-2}) \tag{6}$$

The above expression for $\langle r^2\rangle_{tr}$ may be used as the average mean square radius of the nuclear density in the formula for $\hbar\omega$ given in Daskaloyannis et al. (1983),

$$\hbar\omega = \frac{3}{4}\frac{\hbar^2}{MA}\left[(K+1)\left(A+\frac{1}{3}n\right)+\frac{2}{3}n-2\right][\overline{\langle r^2\rangle} - (\langle r^2\rangle_p + \langle r^2\rangle_n)]^{-1} \tag{7}$$

where $\langle r^2\rangle_p + \langle r^2\rangle_n = 0.659\ \text{fm}^2$ (Chandra and Sauer, 1976).

Expansions of $\langle r^2\rangle_{tr}^{1/2}$, $\langle r^2\rangle_{tr}$, and $\hbar\omega$ in powers of the mass number A (or of the quantity $z/bA^{1/3}$) can be obtained by using, as in Daskaloyannis et al. (1983), the corresponding expansion for c (Elton, 1961, pp. 31, 107):

$$c = bA^{1/3}\left[1-\frac{1}{3}\left(\frac{z}{bA^{1/3}}\right)^2+\frac{1}{81}\left(\frac{z}{bA^{1/3}}\right)^6+\frac{1}{243}\left(\frac{z}{bA^{1/3}}\right)^8+\cdots\right] \tag{8}$$

The following expressions for $\langle r^2\rangle_{tr}^{1/2}$ and $\langle r^2\rangle_{tr}$ result (see also Hornyak, 1975)

$$\langle r^2\rangle_{tr}^{1/2} = \left(\frac{3}{5}\right)^{1/2} bA^{1/3}\left[1+\frac{5}{6}\left(\frac{z}{bA^{1/3}}\right)^2-\frac{69}{72}\left(\frac{z}{bA^{1/3}}\right)^4+\frac{1339}{1296}\left(\frac{z}{bA^{1/3}}\right)^6+\cdots\right] \tag{9}$$

$$\langle r^2\rangle_{tr} = \frac{3}{5}b^2A^{2/3}\left[1+\frac{5}{3}\left(\frac{z}{bA^{1/3}}\right)^2-\frac{11}{9}\left(\frac{z}{bA^{1/3}}\right)^4+\frac{38}{81}\left(\frac{z}{bA^{1/3}}\right)^6+\cdots\right] \tag{10}$$

The above expressions may be compared with the corresponding ones for the Fermi distribution:

$$\langle r^2\rangle_F^{1/2} = \left(\frac{3}{5}\right)^{1/2} bA^{1/3}\left[1+\frac{5}{6}\left(\frac{\pi\alpha}{bA^{1/3}}\right)^2-\frac{7}{24}\left(\frac{\pi\alpha}{bA^{1/3}}\right)^4+\frac{1003}{1296}\left(\frac{\pi\alpha}{bA^{1/3}}\right)^6+\cdots\right] \tag{11}$$

$$\langle r^2\rangle_F=\frac{3}{5}b^2A^{2/3}\left[1+\frac{5}{3}\left(\frac{\pi\alpha}{bA^{1/3}}\right)^2+\frac{1}{9}\left(\frac{\pi\alpha}{bA^{1/3}}\right)^4+\frac{2}{81}\left(\frac{\pi\alpha}{bA^{1/3}}\right)^6+\cdots\right] \tag{12}$$

By using in expression (7) for $\hbar\omega$ the approximation $\overline{\langle r^2\rangle}\simeq\langle r^2\rangle_{tr}$ and also the expression for the highest filled shell K as a function of the mass number A and of the number of valence nucleons n [formula (4) of Daskaloyannis et al. (1983)], we obtain after some algebra the following expansion for $\hbar\omega_{tr}$:

$$\hbar\omega_{tr}=\frac{5}{4}\left(\frac{3}{2}\right)^{1/3}\frac{\hbar^2}{M}b^{-2}A^{-1/3}(1+D_1A^{-2/3}+D_2A^{-4/3}+D_3A^{-5/3}+\cdots) \tag{13}$$

where

$$D_1=\frac{1}{3}\left(\frac{2}{3}\right)^{2/3}-\left(\frac{5}{3}+\frac{\lambda'}{z^2}\right)\left(\frac{z}{b}\right)^2 \tag{14}$$

$$D_2=\left(\frac{2}{3}\right)^{1/3}\left(\frac{2n}{3}-2\right)-\frac{1}{3}\left(\frac{2}{3}\right)^{2/3}\frac{5}{3}\left(\frac{z}{b}\right)^2+\left[\frac{50}{9}\left(\frac{z}{b}\right)^2-\frac{1}{3}\left(\frac{2}{3}\right)^{2/3}\lambda'\right]b^{-2}$$
$$+\lambda'^2b^{-4}+\frac{36}{9}\left(\frac{z}{b}\right)^4 \tag{15}$$

$$D_3=\frac{2n}{9}\left(\frac{2}{3}\right)^{2/3} \tag{16}$$

and

$$\lambda'=-\tfrac{5}{3}(\langle r^2\rangle_p+\langle r^2\rangle_n)\simeq-\tfrac{5}{3}\cdot 0.659=-1.098\ \text{fm}^2 \tag{17}$$

The coefficients D_i $(i=1,2,3)$ may be compared with the coefficients C_i of Daskaloyannis et al. (1983). The expression for C_3 (not given there) may be easily found to be

$$C_3=\frac{2n}{9}\left(\frac{2}{3}\right)^{2/3}=D_3 \tag{18}$$

It is interesting to note the remarkable similarity of the two expansions for $\hbar\omega$.

A final comment is appropriate: In the previous analysis the parameters b and z of the distribution were taken independent of A, this being an approximation. The possibility of the variation of these parameters with A may be considered. This variation will affect the functional relation between c and A and vice versa (see, for example, Preston and Bhadhuri, 1975, p. 119). We assume that these parameters may be approximated by power series in $A^{-1/3}$. This is in fact analogous to the treatment of parameters in the mass formula (Kodama, 1971). We write

$$\begin{aligned} b &= b(A) = b_0 + b_1 A^{-1/3} + b_2 A^{-2/3} + \cdots \\ z &= z(A) = z_0 + z_1 A^{-1/3} + z_2 A^{-2/3} + \cdots \end{aligned} \tag{19}$$

and use appropriate truncated expressions of these expansions. The b_i and z_i in these truncated expressions are treated as fitting parameters.

The expansions of $\langle r^2\rangle_{\rm tr}^{1/2}$, $\langle r^2\rangle_{\rm tr}$, and $\hbar\omega_{\rm tr}$ are now modified as follows:

$$\begin{aligned} \langle r^2\rangle_{\rm tr}^{1/2} = \left(\frac{3}{5}\right)^{1/2} b_0 A^{1/3} \Bigg[1 + \frac{b_1}{b_0} A^{-1/3} + \left(\frac{b_2}{b_0} + \frac{5}{6}\frac{z_0^2}{b_0^2}\right) A^{-2/3} \\ + \left(\frac{b_3}{b_0} + \frac{5}{3}\frac{z_0 z_1}{b_0^2} - \frac{5}{6}\frac{z_0^2 b_1}{b_0^3}\right) A^{-1} + \cdots \Bigg] \end{aligned} \tag{20}$$

$$\begin{aligned} \langle r^2\rangle_{\rm tr} = \frac{3}{5} b_0^2 A^{2/3} \Bigg[1 + 2\frac{b_1}{b_0} A^{-1/3} + \left(2\frac{b_2}{b_0} + \frac{3b_1^2 + 5z_0^2}{3b_0^2}\right) A^{-2/3} \\ + \left(2\frac{b_3}{b_0} + 2\frac{b_1 b_2}{b_0^2} + \frac{10}{3}\frac{z_0 z_1}{b_0^2}\right) A^{-1} + \cdots \Bigg] \end{aligned} \tag{21}$$

$$\begin{aligned} \hbar\omega_{\rm tr} = \frac{5}{4}\left(\frac{3}{2}\right)^{1/3} \frac{\hbar^2}{M} b_0^{-2} A^{-1/3} \Bigg(1 - 2\frac{b_1}{b_0} A^{-1/3} + \Bigg[\frac{1}{3}\left(\frac{2}{3}\right)^{2/3} - 2\frac{b_2}{b_0} \\ + \left(3b_1^2 - \frac{5}{3} z_0^2 - \lambda'\right) \frac{1}{b_0^2} \Bigg] A^{-2/3} + \Bigg\{ -2\left[b_3 + \frac{1}{3}\left(\frac{2}{3}\right)^{2/3} b_1 \right] \frac{1}{b_0} \\ + \left(6b_1 b_2 - \frac{10}{3} z_0 z_1\right) \frac{1}{b_0^2} + \Bigg(-4b_1^3 \\ + \frac{20}{3} z_0^2 b_1 + 4b_1\lambda' \Bigg) \frac{1}{b_0^3} \Bigg\} A^{-1} + \cdots \Bigg) \end{aligned} \tag{22}$$

It is seen that additional terms appear in comparison with the previous expansions. It is also interesting to note that if only the two first terms in the expansion of $\langle r^2\rangle_{\rm tr}^{1/2}$ are kept, the structure of the resulting formula for the rms radius is in agreement with the known empirical expression. The powers of A in the expansion of $\hbar\omega$ are now the same as those appearing if the expressions of $\hbar\omega$ of Eder (1968) and Hornyak (1975) are expanded [see also Blomqvist and Molinari (1968) and Bertsch (1972), where the first two terms of the expansion were kept]. It is also seen that there are now more leading terms of the expansion of $\hbar\omega$, which are independent of the number of the valence nucleons.

3. NUMERICAL RESULTS AND DISCUSSION

The values of the parameters ρ_0 and z needed for the computation of $\hbar\omega$ are obtained in the same way as in Daskaloyannis et al. (1983): (a) by fitting the theoretical expression for the rms radius (6) to the experimental rms radii of the charge distributions of individual nuclei (Jager et al., 1974; and personal communication of updated compilations) in the region $12 \leq A \leq 208$ and (b) by fitting the theoretical expression for $\hbar\omega$, (7), to the "experimental" values of $\hbar\omega : \hbar\omega_{\langle r^2\rangle_{\text{exp}}}$ in the region $12 \leq A \leq 40$. These "experimental" values reproduce by means of (7) the experimental values of the rms radii of the charge distributions of the relevant nuclei, as these are determined from the elastic electron scattering experiments. Both fittings are unweighted least-square fittings, as in Daskaloyannis et al. (1983).

The fitting of the "first kind" (described above) gave the best fit values $\rho_0 = 0.167\ \text{fm}^{-3}$ and $z = 1.863$ fm, while that of the "second kind" gave $\rho_0 = 0.131\ \text{fm}^{-3}$ and $z = 1.366$ fm. The results of the computations are shown in Figures 2 and 3, where we also plot the corresponding curves obtained with the Fermi distribution from Daskaloyannis et al. (1983) for the sake of comparison. An improvement is observed in Figure 1 with the trapezoidal density for small values of A. This should be attributed to the fact that the

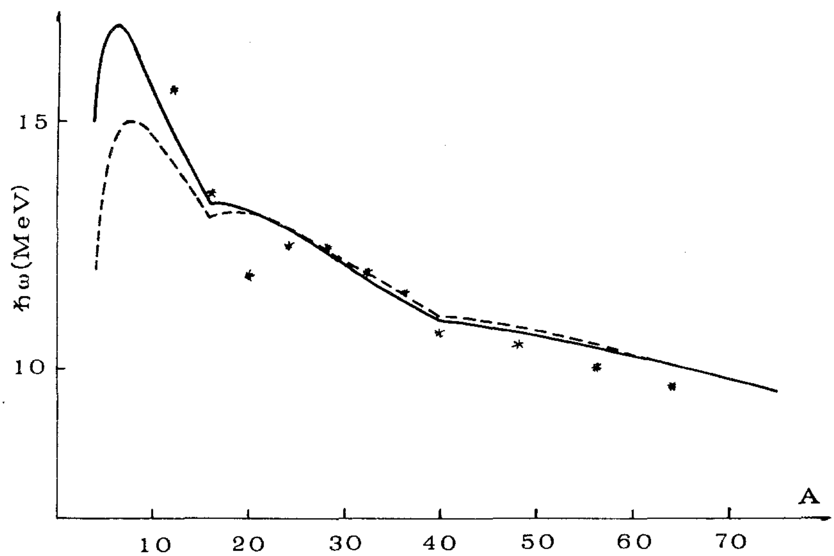

Fig. 2. (—) Variation of $\hbar\omega$ with the mass number A, using a trapezoidal distribution with $\rho_0 = 0.167\ \text{fm}^{-3}$ and $z = 1.863$ fm determined by fitting the theoretical expression $\langle r^2\rangle_{\text{tr}}^{1/2}$ to experimental rms radii of nuclei. (- -) The curve corresponding to the Fermi distribution. Also shown are the values of $\hbar\omega_{\langle r^2\rangle_{\text{exp}}}$ (see text).

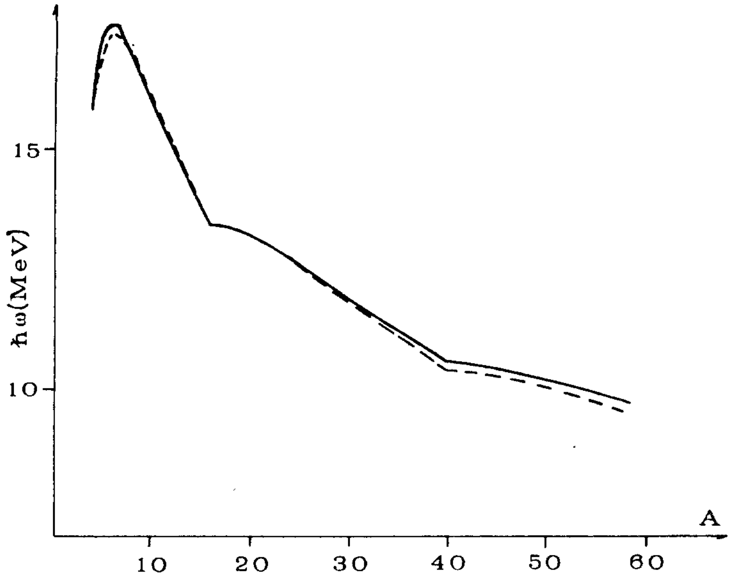

Fig. 3. The variation of $\hbar\omega$ with A by fitting the theoretical expression of $\hbar\omega$ with $\langle \bar{r}^2\rangle = \langle r^2\rangle_{tr}$ to $\hbar\omega_{\langle r^2\rangle_{exp}}$ (see text). (—) Result for the trapezoidal distribution; (- -) the Fermi distribution.

integrals for this density are calculated exactly. The differences, however, between the two curves in Figure 2 are very small.

We report our numerical results using the expansions (19). We consider as a first approximation only two fitting parameters, b_0 and b_1, that is, we take $b_2 = b_3 = \cdots = 0$ and $z = 0$. In this case we have the following expression for the rms radius:

$$\langle r^2\rangle^{1/2} = f_1 A^{1/3} + f_2, \qquad f_1 = \left(\frac{3}{5}\right)^{1/2} b_0, \quad f_2 = \frac{b_1}{b_0} f_1 \tag{23}$$

which, as was pointed out previously, is of the same form as the empirical expression that has appeared rather often in the literature (Eder, 1968; Hornyak, 1975; Green, 1956; Green et al., 1968, p. 77; Hofstadter and Collard, 1967). It is therefore seen that this type of expression may follow from a density distribution, as, for example, a uniform one, if an expansion of type (19) is assumed for the parameter b (which is related to the central density) and a truncation is made. The (unweighted least square) fitting of expression (23) to the $\langle r^2\rangle_{exp}^{1/2}$ of nuclei in the region $12 \leq A \leq 208$ gives

$$\langle r^2\rangle^{1/2} = 0.81 A^{1/3} + 0.68 \tag{24}$$

This may be compared with the corresponding expression of Hofstadter and Collard (1967):

$$\langle r^2\rangle^{1/2} = 0.82A^{1/3} + 0.58 \tag{25}$$

The fitting of the "second kind" with two parameters b_0 and b_1 gives

$$\hbar\omega = 42.4A^{-1/3} - 18.4A^{-2/3} \tag{26}$$

This expression may be compared with the corresponding expression of Blomqvist and Molinari (1968):

$$\hbar\omega = 45A^{-1/3} - 25A^{-2/3} \tag{27}$$

It should be noted that if we use three fitting parameters b_0, b_1, and z_0 we obtain for the leading terms in the expansion of $\langle r^2\rangle_{\mathrm{tr}}^{1/2}$

$$\langle r^2\rangle_{\mathrm{tr}}^{1/2} = 0.83A^{1/3} + 0.47 + 0.60A^{-1/3} \tag{28}$$

The corresponding expression for $\hbar\omega_{\mathrm{tr}}$ is

$$\langle\hbar\omega\rangle_{\mathrm{tr}} = 51.2A^{-1/3} - 58.0A^{-2/3} + 64.0A^{-1} \tag{29}$$

It has not been possible, however, to obtain an expansion of $\hbar\omega_{\mathrm{tr}}$ with reasonable coefficients in the case of the second kind of fitting with three adjustable parameters.

We note finally that the results of this paper and of other work pertaining to this problem might be useful in comparing the expansion in powers of the mass number of the oscillator spacing for a nucleon in a nucleus with the corresponding one for a Λ-particle in a hypernucleus, which has been a matter of recent investigation (Daskaloyannis et al., 1985; see also Daskaloyannis et al., 1984; Lalazissis et al., 1987).

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